LANCASTER UNIVERSITY

YFAW 2026

The Class of Banach Lattices is not primary

Forum of Mathematics, Sigma (2026)

Antonio H. Acuaviva

Primariness of a Banach space

Given a Banach space, how might we split $X = Y \oplus Z$?

$X = \ell_2$
"Basic object"

$X = \ell_2 \oplus \ell_1$
"Composite object"

Definition 1. Let X be a Banach space. We say that X is *primary* if whenever $X = Y \oplus Z$, then $X \sim Y$ or $X \sim Z$.

Definition 1. Let X be a Banach space. We say that X is *primary* if whenever $X = Y \oplus Z$, then $X \sim Y$ or $X \sim Z$.

Examples: Most of the classical Banach spaces!

(a) $X = \ell_p$, $1 \leq p < \infty$ (Pełczyński, 1960), $p = \infty$ (Lindenstrauss, 1967),

(b) $X = L_p[0,1]$, $1 < p < \infty$, (D. Alspach, P. Enflo, and E. Odell, 1977), $p = 1$ (Enflo, via Maurey 1974 & Enflo and Starbird, 1979),

(c) $X = C[0,1]$ (Lindenstrauss and Pełczyński, 1971),

(d) $X = C(\alpha)$ (Alspach and Benyamini, 1977).

(e) Many others!

Primariness of a “Class” of Banach space

“Class” as a “collection” of Banach spaces having some given properties

Examples:

(a) Separable spaces, $\mathscr{X} = \{X : X \text{ is separable}\}$.

(b) Reflexive spaces, $\mathscr{X} = \{X : X \text{ is reflexive}\}$.

(c) Primary spaces, $\mathscr{X} = \{X : X \text{ is primary}\}$.

(d) $C(K)$-spaces, $\mathscr{X} = \{X : X \sim C(K), \text{ for some compact Hausdorff } K\}$.

(e) **Banach lattices**, $\mathscr{X} = \{X : X \sim L, \text{ for some Banach lattice } L\}$.

Given a Banach space $X \in \mathscr{X}$, how might we split $X = Y \oplus Z$?

Primariness of a "Class" of Banach space

Definition 2. Let $\mathscr{X}$ be a class of Banach spaces. We say that $\mathscr{X}$ is *primary* if whenever $X \in \mathscr{X}$ and $X = Y \oplus Z$, then $Y \in \mathscr{X}$ or $Z \in \mathscr{X}$.

Examples:

(a) Separable spaces, $\mathscr{X} = \{X : X \text{ is separable}\}$. $\longrightarrow$ Primary

(b) Reflexive spaces, $\mathscr{X} = \{X : X \text{ is reflexive}\}$. $\longrightarrow$ Primary

(c) Primary spaces, $\mathscr{X} = \{X : X \text{ is primary}\}$. $\longrightarrow$ Primary

(d) $C(K)$-spaces, $\mathscr{X} = \{X : X \sim C(K), \text{ for some compact Hausdorff } K\}$.

(e) **Banach lattices**, $\mathscr{X} = \{X : X \sim L, \text{ for some Banach lattice } L\}$.

Banach lattices

Definition 3. A *Banach lattice* is a real Banach space X equipped with an order $\leq$ such that X is a vector lattice and the norm is monotone: if $|x| \leq |y|$, then $\|x\| \leq \|y\|$.

Example: Continuous functions on the unit interval, i.e $X = C[0,1]$, $\|f\| = \sup_{t\in[0,1]} |f(t)|$

- Order (pointwise): $f \leq g$ iff $f(t) \leq g(t)$ for all $t \in [0,1]$
- Vector lattice: given $f, g \in C[0,1]$ we can define $h = \sup\{f,g\}$ and $r = \inf\{f,g\}$ (pointwise!)
- Monotone: $|f| \leq |g| \implies \|f\| = \sup_{t\in[0,1]} |f(t)| \leq \sup_{t\in[0,1]} |g(t)| = \|g\|$

Examples: ℓ_p and $L_p(\mu)$ for $1 \leq p \leq \infty$ and $C(K)$-spaces.

Main Theorem

Theorem 1.1. *There exist a compact Hausdorff space L and Banach spaces X and $\tilde{X}$ such that*

$$C(L) \simeq X \oplus \tilde{X},$$

and neither X nor $\tilde{X}$ is isomorphic to a Banach lattice.

Corollary 1. The class of Banach lattices and the class of $C(K)$-spaces is not primary.

Theorem 1 (Milutin, Lindenstrauss–Pełczyński and Alspach–Benyamini)**.** The class of *separable* $C(K)$-spaces is primary.

The proof of the theorem

Theorem 2 (Plebanek and Salguero Alarcón; 2023)**.** There exist compact Hausdorff spaces K and L such that

$$C(L) = C(K) \oplus \mathrm{PS}_2$$

and PS_2 fails to be a isomorphic to a $C(K)$-space.

Theorem 3 (De Hevia, Martínez-Cervantes, Salguero-Alarcón, and Tradacete; 2025)**.** The space PS_2 fails to be isomorphic to a Banach lattice.

The proof of the theorem, preliminary

Theorem 2 (Plebanek and Salguero Alarcón; 2023)**.** There exist compact Hausdorff spaces K and L such that

$$C(L) = C(K) \oplus \mathrm{PS}_2$$

and PS_2 fails to be a isomorphic to a $C(K)$-space.

Key ideas of the proof.

- The fact that PS_2 is isomorphic to a $C(K)$-space (similarly, to a Banach lattice) can be characterized by *continuum many* conditions.
- We can construct compact Hausdorff spaces L by a transfinite construction with *continuum many* steps, involving *continuum many* choices in each step.
- At each stage, the choice is made so as to ensure that PS_2 is *not* isomorphic to a $C(K)$-space (similarly, to a Banach lattice).

The proof of the main theorem

Theorem 1.1. *There exist a compact Hausdorff space L and Banach spaces X and $\tilde{X}$ such that*

$$C(L) \simeq X \oplus \tilde{X},$$

and neither X nor $\tilde{X}$ is isomorphic to a Banach lattice.

$$\left.\begin{aligned} C(L) &= C(K) \oplus \mathrm{PS}_2 \\ C(\widetilde{L}) &= C(\widetilde{K}) \oplus \widetilde{\mathrm{PS}_2} \end{aligned}\right\} C(L \sqcup \widetilde{L}) = C(L) \oplus C(\widetilde{L}) = \underbrace{(C(\widetilde{K}) \oplus \mathrm{PS}_2)}_{X} \oplus \underbrace{(C(K) \oplus \widetilde{\mathrm{PS}_2})}_{\tilde{X}}$$

Thanks!

Questions?