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Factorizations and norms on Calkin algebras of $C(K)$ -spaces

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Definition. An algebra $\mathcal{A}$ (over a field $\mathbb{K}$) is a vector space equipped with a bilinear product. A norm $\|\cdot\|$ on an algebra $\mathcal{A}$ is a vector norm making multiplication submultiplicative, i.e.

(a) $\|a + b\| \leq \|a\| + \|b\|$, for all $a, b \in \mathcal{A}$.

(b) $\|\lambda a\| = |\lambda| \|a\|$, for all $\lambda \in \mathbb{K}$, $a \in \mathcal{A}$.

(c) $\|ab\| \leq \|a\| \|b\|$ for $a, b \in \mathcal{A}$.

$\mathcal{A}$

$\|\cdot\|$

$$C(K) = \{f : K \rightarrow \mathbb{K}, f \text{ continuous} \}$$

$$\mathcal{B}(X)$$

$$\mathcal{C}(X) = \mathcal{B}(X)/\mathcal{K}(X)$$

$$\|f\| = \sup_{k \in K} |f(k)|$$

$$\|T\| = \sup_{x \in B_X} \|Tx\|$$

$$\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}(X)\}$$



Definition. Let $\|\cdot\|$ be a norm on an algebra $\mathcal{A}$, then we say that

- (a) $\|\cdot\|$ is *maximal* if the topology determined by $\|\cdot\|$ is largest/finest of any other algebra-norm-induced topology, i.e. for any other norm $\|\!\|\!\cdot\!\|\!$, there exists C such that $\|\!\|\!\cdot\!\|\! \leq C \|\cdot\|$.
- (b) $\|\cdot\|$ is *minimal* if the topology determined by $\|\cdot\|$ is smallest/weakest of any other algebra-norm-induced topology, i.e. for any other norm $\|\!\|\!\cdot\!\|\!$, there exists c such that $c \|\cdot\| \leq \|\!\|\!\cdot\!\|\!$.

Definition. We say that an algebra $\mathcal{A}$ admits a *unique algebra norm* if all the algebra norms on $\mathcal{A}$ generate the same topology.

$$\text{Maximal} + \text{Minimal} \iff \text{Unique norm}$$



Theorem (Eidelheit, 1940). Let X be a Banach space, and let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(X)$ which contains the ideal of finite-rank operators. Then the operator norm on $\mathcal{A}$ is minimal.

Theorem (Meyer, 1995). The Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ has a unique algebra norm for $X = c_0$ and $X = \ell_p$, where $1 \leq p < \infty$.

Theorem (Ware, 2014). The Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ has a unique algebra norm for many Banach spaces X , including

- (a) every finite direct sum of spaces from the family $\{c_0\} \cup \{\ell_p : 1 \leq p < \infty\}$;
- (b) the infinite direct sums $(\bigoplus_{n \in \mathbb{N}} \ell_p^n)_{c_0}$ and $(\bigoplus_{n \in \mathbb{N}} \ell_p^n)_{\ell_q}$
- (c) the Tsirelson space T and its dual T^* ;
- (d) the quasi-reflexive James spaces J_p for $1 < p < \infty$.



Theorem (Johnson and Phillips, to appear). The Calkin Algebra $\mathcal{B}(L_p[0, 1])/\mathcal{K}(L_p[0, 1])$ has a unique algebra norm whenever $1 < p < \infty$.

Theorem (Arnott and Laustsen, 2023). The Calkin Algebra $\mathcal{B}(C_0(K_{\mathcal{A}}))/\mathcal{K}(C_0(K_{\mathcal{A}}))$ has a unique algebra norm, where $K_{\mathcal{A}}$ is any Mrówka space chosen such that $C_0(K_{\mathcal{A}})$ admits “few operators”.

What can we say about the Calkin algebra of $C(K)$?

What about when K is an ordinal interval?

More generally, K scattered space!



Algebraic relations $\implies$ Norm constraints

$$I \longrightarrow \|I\| \geq 1$$

$$P, P^n = P \longrightarrow \|P\| \geq 1$$

$$P, \exists U \text{ such that } PU = U \longrightarrow \|P\| \geq 1$$

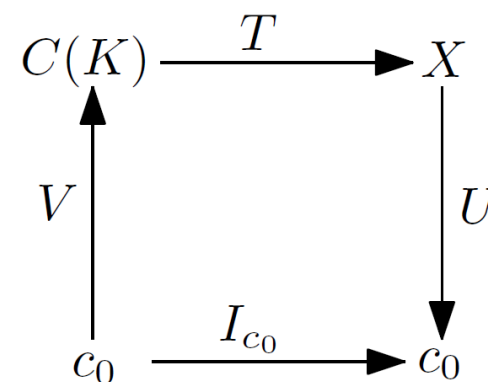
Norm properties $\implies$ Algebraic relations $\implies$ Norms constraints

Theorem (Pełczyński, 1962). Any non-weakly compact operator $T : C(K) \rightarrow X$ fixes a copy of c_0 .

Theorem (Pełczyński, 1962; factorization form by Arnett and Laustsen, 2023). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^* -sequentially compact, and let K be a compact Hausdorff space.

Then, for any non-weakly compact operator $T : C(K) \rightarrow X$, there exist $V \in \mathcal{B}(c_0, C(K))$ and $U \in \mathcal{B}(X, c_0)$ such that

$$UTV = I_{c_0}.$$





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Theorem (Quantitative version). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^* -sequentially compact, and let K be a scattered, compact Hausdorff space.

Then, for any $\varepsilon > 0$ and any non-compact operator $T : C(K) \rightarrow X$, there exist $V \in \mathcal{B}(c_0, C(K))$ and $U \in \mathcal{B}(X, c_0)$ such that

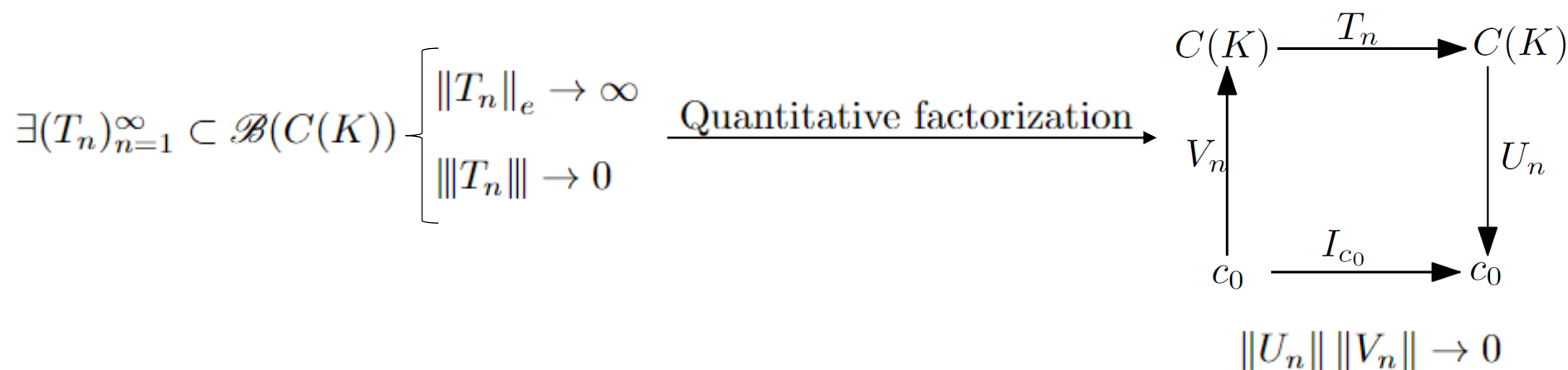
$$UTV = I_{c_0}, \quad \|U\| \|V\| < \frac{2 + \varepsilon}{\|T\|_e}.$$

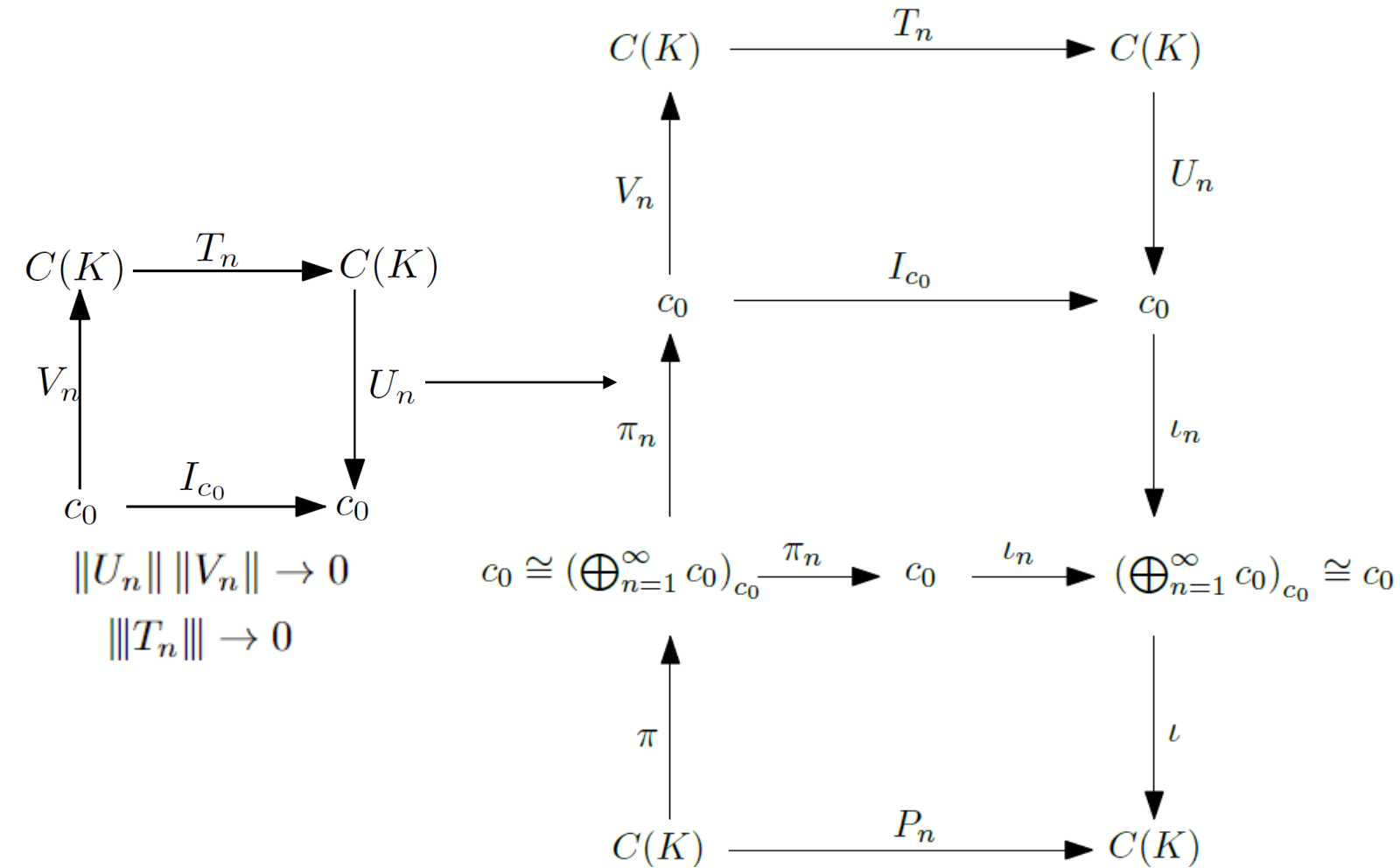
Theorem. Let K be a scattered, compact Hausdorff space and consider the Calkin algebra $\mathcal{B}(C(K))/\mathcal{K}(C(K))$. Then the essential norm $\|\cdot\|_e$ is minimal.

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Proof (Sketch): Minimal: $\forall \|\cdot\| \xrightarrow{\exists c > 0} c \|\cdot\|_e \leq \|\cdot\|$

Not Minimal: $\exists \|\cdot\| \xrightarrow{\forall c > 0} c \|\cdot\|_e \not\leq \|\cdot\|$





$$\begin{aligned}
 & \|U_n\| \|V_n\| \rightarrow 0 \\
 & \downarrow \\
 & U = \sum_{n=1}^{\infty} \iota_n U_n \\
 & V = \sum_{n=1}^{\infty} V_n \pi_n \pi \\
 & \downarrow \\
 & P_n U T_n V P_n = P_n \quad \text{for all } n \in \mathbb{N}, \\
 & \downarrow \\
 & 0 < \frac{1}{\|U\| \|V\|} \leq \|P_n\| \|T_n\| \rightarrow 0 \\
 & \downarrow \\
 & \text{Contradiction} \quad \square
 \end{aligned}$$



Corollary. Let K be a scattered, compact Hausdorff space. Then the Calkin algebra $\mathcal{B}(C(K))/\mathcal{K}(C(K))$ admits a unique complete algebra norm.

Theorem (Ogden, 1996). The essential norm is maximal in the Calkin algebra $\mathcal{B}(C[0, \omega_\eta])/\mathcal{K}(C[0, \omega_\eta])$, where ω_η is the first ordinal of cardinality $\aleph_\eta$ for an ordinal η .

Corollary. The Calkin algebra $\mathcal{B}(C[0, \omega_\eta])/\mathcal{K}(C[0, \omega_\eta])$ admits a unique algebra norm.



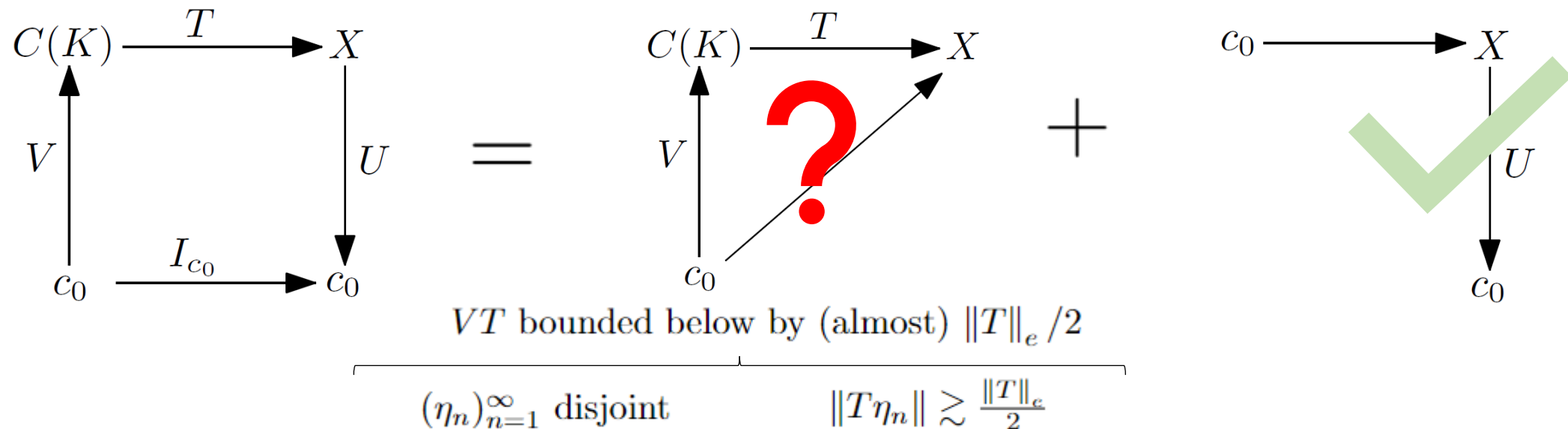
Thanks!

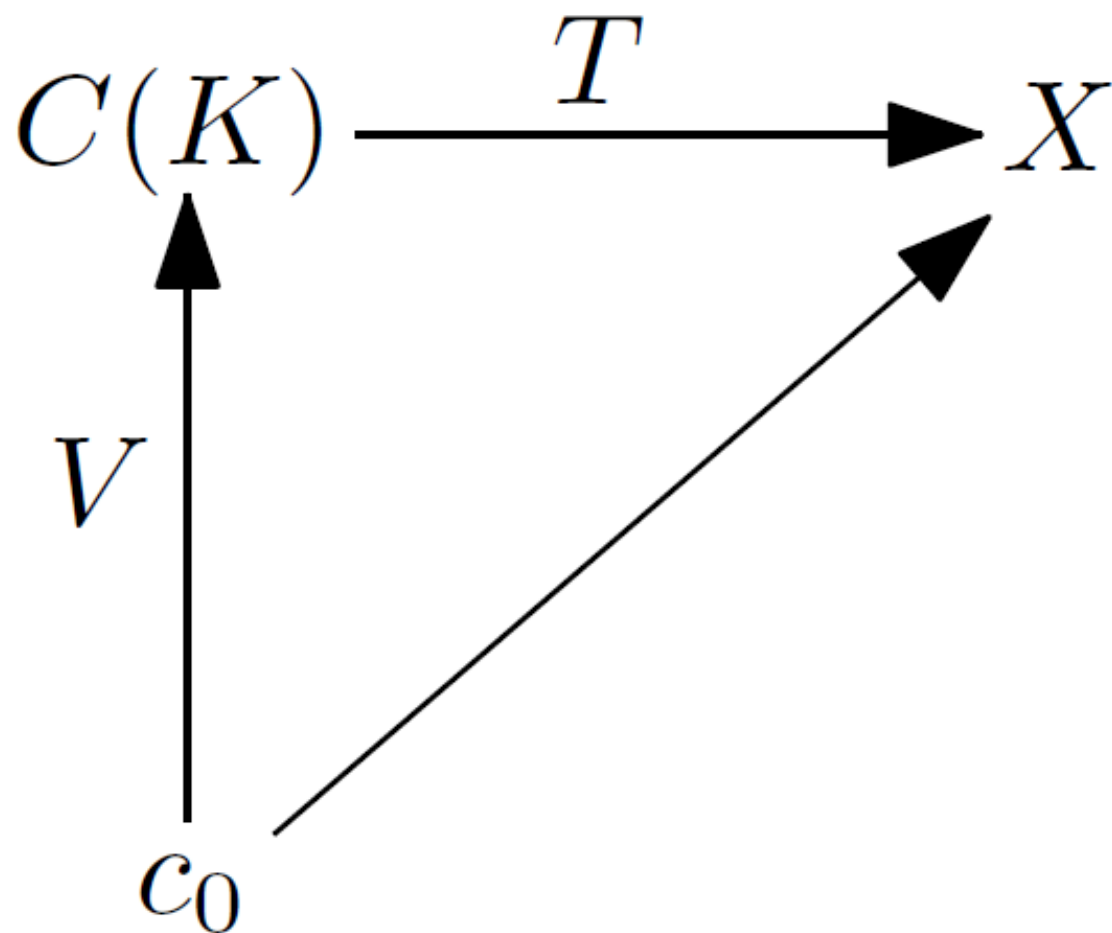
Questions?

Theorem (Quantitative version). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^* -sequentially compact, and let K be a scattered, compact Hausdorff space.

Then, for any $\varepsilon > 0$ and any non-compact operator $T : C(K) \rightarrow X$, there exist $V \in \mathcal{B}(c_0, C(K))$ and $U \in \mathcal{B}(X, c_0)$ such that

$$UTV = I_{c_0}, \quad \|U\| \|V\| < \frac{2 + \varepsilon}{\|T\|_e}.$$





VT bounded below by (almost) $\|T\|_e / 2$

$(\eta_n)_{n=1}^\infty$ disjoint

$$\|T\eta_n\| \gtrsim \frac{\|T\|_e}{2}$$

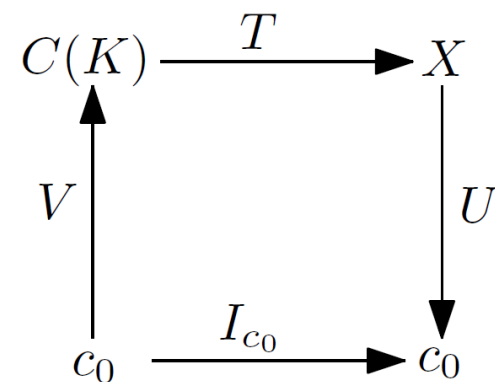
- Compact operators are nice to work with!
- $C(K)^* \cong \ell^1(K)$, nice space!



Theorem (Pełczyński, 1962; factorization form by Arnott and Laustsen, 2023). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^* -sequentially compact, and let K be a compact Hausdorff space.

Then, for any non-weakly compact operator $T : C(K) \rightarrow X$, there exist $V \in \mathcal{B}(c_0, C(K))$ and $U \in \mathcal{B}(X, c_0)$ such that

$$UTV = I_{c_0}.$$



Definition. Given $T : C(K) \rightarrow X$ we define

$$\Delta(T) = \sup_{(f_n)} \limsup_{n \rightarrow \infty} \|Tf_n\|,$$

where the supremum is taken over all families of disjoint functions (f_n) .

Remark: $\Delta(T) = 0$ if and only if T is weakly compact.



Theorem (“Trivial” quantitative version). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^* -sequentially compact, and let K be a compact Hausdorff space.

Then, for any $\varepsilon > 0$ and any non-weakly compact operator $T : C(K) \rightarrow X$, there exist $V \in \mathcal{B}(c_0, C(K))$ and $U \in \mathcal{B}(X, c_0)$ such that

$$UTV = I_{c_0}, \quad \|U\| \|V\| < \frac{1 + \varepsilon}{\Delta(T)}.$$

$$K = [0, 1]$$

Question 1. Is $\Delta(T) \sim \|T\|_{we}$ for $K = [0, 1]$?

Question 2. Is $\|T\|_{we}$ dual, i.e. does there exist c such that

$$\|T\|_{we} \geq \|T^*\|_{we} \geq c \|T\|_{we}.$$



Questions?