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Factorizations and norms on Calkin algebras of C(K)-spaces

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Algebras and Normed algebras

Definition. An algebra $\mathcal{A}$ (over a field $\mathbb{K}$) is a vector space equipped with a bilinear product. A norm $\|\cdot\|$ on an algebra $\mathcal{A}$ is a vector norm making multiplication submultiplicative, i.e.

(a) $\|a+b\| \leq \|a\| + \|b\|$, for all $a, b \in \mathcal{A}$.

(b) $\|\lambda a\| = |\lambda|\,\|a\|$, for all $\lambda \in \mathbb{K}$, $a \in \mathcal{A}$.

(c) $\|ab\| \leq \|a\|\,\|b\|$ for $a, b \in \mathcal{A}$.

$$\mathcal{A}$$

$$C(K) = \{f : K \to \mathbb{K}, f \text{ continuous } \}$$

$$\mathscr{B}(X)$$

$$\mathscr{C}(X) = \mathscr{B}(X)/\mathscr{K}(X)$$

$$\|\cdot\|$$

$$\|f\| = \sup_{k\in K} |f(k)|$$

$$\|T\| = \sup_{x\in B_X} \|Tx\|$$

$$\|T\|_e = \inf\{\|T-S\| : S \in \mathscr{K}(X)\}$$

Uniqueness-of-norm-problem

Definition. Let $\|\cdot\|$ be a norm on an algebra $\mathcal{A}$, then we say that

(a) $\|\cdot\|$ is *maximal* if the topology determined by $\|\cdot\|$ is largest/finest of any other algebra-norm-induced topology, i.e. for any other norm $|||\cdot|||$, there exists C such that $|||\cdot||| \leq C\,\|\cdot\|$.

(b) $\|\cdot\|$ is *minimal* if the topology determined by $\|\cdot\|$ is smallest/weakest of any other algebra-norm-induced topology, i.e. for any other norm $|||\cdot|||$, there exists c such that $c\,\|\cdot\| \leq |||\cdot|||$.

Definition. We say that an algebra $\mathcal{A}$ admits a *unique algebra norm* if all the algebra norms on $\mathcal{A}$ generate the same topology.

$$\text{Maximal} + \text{Minimal} \iff \text{Unique norm}$$

Some known-results

Theorem (Eidelheit, 1940). Let X be a Banach space, and let $\mathcal{A}$ be a subalgebra of $\mathscr{B}(X)$ which contains the ideal of finite-rank operators. Then the operator norm on $\mathcal{A}$ is minimal.

Theorem (Meyer, 1995). The Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ has a unique algebra norm for $X = c_0$ and $X = \ell_p$, where $1 \leq p < \infty$.

Theorem (Ware, 2014). The Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ has a unique algebra norm for many Banach spaces X, including

(a) every finite direct sum of spaces from the family $\{c_0\} \cup \{\ell_p : 1 \leq p < \infty\}$;

(b) the infinite direct sums $\left(\bigoplus_{n\in\mathbb{N}} \ell_p^n\right)_{c_0}$ and $\left(\bigoplus_{n\in\mathbb{N}} \ell_p^n\right)_{\ell_q}$

(c) the Tsirelson space T and its dual T^*;

(d) the quasi-reflexive James spaces J_p for $1 < p < \infty$.

Some known-results

Theorem (Johnson and Phillips, to appear). The Calkin Algebra $\mathscr{B}(L_p[0,1])/\mathscr{K}(L_p[0,1])$ has a unique algebra norm whenever $1 < p < \infty$.

Theorem (Arnott and Laustsen, 2023). The Calkin Algebra $\mathscr{B}(C_0(K_{\mathcal{A}}))/\mathscr{K}(C_0(K_{\mathcal{A}}))$ has a unique algebra norm, where $K_{\mathcal{A}}$ is any Mrówka space chosen such that $C_0(K_{\mathcal{A}})$ admits "few operators".

What can we say about the Calkin algebra of $C(K)$?

What about when K is an ordinal interval?
More generally, K scattered space!

Algebraic relations and factorizations

$$\text{Algebraic relations} \implies \text{Norm constrains}$$

$$I \longrightarrow \|I\| \geq 1$$

$$P,\ P^n = P \longrightarrow \|P\| \geq 1$$

$$P,\ \exists U \text{ such that } PU = U \longrightarrow \|P\| \geq 1$$

$$\text{Norm properties} \implies \text{Algebraic relations} \implies \text{Norms constrains}$$

Theorem (Pełczyński, 1962). Any non-weakly compact operator $T : C(K) \to X$ fixes a copy of c_0.

Theorem (Pełczyński, 1962; factorization form by Arnott and Laustsen, 2023). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^*-sequentially compact, and let K be a compact Hausdorff space.

Then, for any non-weakly compact operator $T : C(K) \to X$, there exist $V \in \mathscr{B}(c_0, C(K))$ and $U \in \mathscr{B}(X, c_0)$ such that

$$UTV = I_{c_0}.$$

$$\begin{tikzcd} C(K) \arrow[r, "T"] & X \arrow[d, "U"] \\ c_0 \arrow[u, "V"] \arrow[r, "I_{c_0}"] & c_0 \end{tikzcd}$$

Quantitative factorization

Theorem (Pełczyński, 1962; factorization form by Arnott and Laustsen, 2023)**.** Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^*-sequentially compact, and let K be a compact Hausdorff space.

Then, for any non-weakly compact operator $T : C(K) \to X$, there exist $V \in \mathscr{B}(c_0, C(K))$ and $U \in \mathscr{B}(X, c_0)$ such that

$$UTV = I_{c_0}.$$

Theorem (Quantitative version)**.** Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^*-sequentially compact, and let K be a **scattered**, compact Hausdorff space.

Then, for any $\varepsilon > 0$ and any non-compact operator $T : C(K) \to X$, there exist $V \in \mathscr{B}(c_0, C(K))$ and $U \in \mathscr{B}(X, c_0)$ such that

$$UTV = I_{c_0}, \quad \|U\| \, \|V\| < \frac{2+\varepsilon}{\|T\|_e}.$$

Theorem. Let K be a scattered, compact Hausdorff space and consider the Calkin algebra $\mathscr{B}(C(K))/\mathscr{K}(C(K))$. Then the essential norm $\|\cdot\|_e$ is minimal.

Minimality of the essential norm

Theorem. Let K be a scattered, compact Hausdorff space and consider the Calkin algebra $\mathscr{B}(C(K))/\mathscr{K}(C(K))$. Then the essential norm $\|\cdot\|_e$ is minimal.

Proof (Sketch):

$$\text{Minimal:}\quad \forall |||\cdot||| \xrightarrow{\exists\, c>0} c\|\cdot\|_e \leq |||\cdot|||$$

$$\text{Not Minimal:}\quad \exists |||\cdot||| \xrightarrow{\forall c>0} c\|\cdot\|_e \nleq |||\cdot|||$$

$$\exists (T_n)_{n=1}^{\infty} \subset \mathscr{B}(C(K)) \begin{cases} \|T_n\|_e \to \infty \\ |||T_n||| \to 0 \end{cases} \xrightarrow{\text{Quantitative factorization}}$$

$$\begin{tikzcd} C(K) \arrow[r, "T_n"] \arrow[d, "U_n"'] & C(K) \\ c_0 \arrow[r, "I_{c_0}"] & c_0 \arrow[u, "V_n"'] \end{tikzcd}$$

$$\|U_n\|\,\|V_n\| \to 0$$

Minimality of the essential norm

$$
\begin{tikzcd}
C(K) \arrow[r, "T_n"] \arrow[d, phantom] & C(K) \arrow[d, "U_n"] \\
c_0 \arrow[u, "V_n"] \arrow[r, "I_{c_0}"] & c_0
\end{tikzcd}
$$

$$\|U_n\|\,\|V_n\| \to 0$$

$$|||T_n||| \to 0$$

$$
\begin{tikzcd}
C(K) \arrow[rr, "T_n"] & & C(K) \arrow[d, "U_n"] \\
c_0 \arrow[u, "V_n"] \arrow[rr, "I_{c_0}"] & & c_0 \arrow[d, "\iota_n"] \\
c_0 \cong \left(\bigoplus_{n=1}^{\infty} c_0\right)_{c_0} \arrow[u, "\pi_n"] \arrow[r, "\pi_n"] & c_0 \arrow[r, "\iota_n"] & \left(\bigoplus_{n=1}^{\infty} c_0\right)_{c_0} \cong c_0 \arrow[d, "\iota"] \\
C(K) \arrow[u, "\pi"] \arrow[rr, "P_n"] & & C(K)
\end{tikzcd}
$$

$$\|U_n\|\,\|V_n\| \to 0$$

$$\downarrow$$

$$U = \sum_{n=1}^{\infty} \iota_n U_n$$

$$V = \sum_{n=1}^{\infty} V_n \pi_n \pi$$

$$\downarrow$$

$$P_n U T_n V P_n = P_n \quad \text{for all } n \in \mathbb{N},$$

$$\downarrow$$

$$0 < \frac{1}{|||U|||\,|||V|||} \leq |||P_n|||\,|||T_n||| \to 0$$

$$\downarrow$$

Contradiction

$\square$

Some immediate consequences

Corollary. Let K be a scattered, compact Hausdorff space. Then the Calkin algebra $\mathscr{B}(C(K))/\mathscr{K}(C(K))$ admits a unique complete algebra norm.

Theorem (Ogden, 1996)**.** The essential norm is maximal in the Calkin algebra $\mathscr{B}(C[0,\omega_\eta])/\mathscr{K}(C[0,\omega_\eta])$, where ω_η is the first ordinal of cardinality $\aleph_\eta$ for an ordinal η.

Corollary. The Calkin algebra $\mathscr{B}(C[0,\omega_\eta])/\mathscr{K}(C[0,\omega_\eta])$ admits a unique algebra norm.

Questions?

Questions?

An overview: quantitative factorization

Theorem (Quantitative version). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^*-sequentially compact, and let K be a **scattered**, compact Hausdorff space.

Then, for any $\varepsilon > 0$ and any non-compact operator $T : C(K) \to X$, there exist $V \in \mathscr{B}(c_0, C(K))$ and $U \in \mathscr{B}(X, c_0)$ such that

$$UTV = I_{c_0}, \quad \|U\|\,\|V\| < \frac{2+\varepsilon}{\|T\|_e}.$$

$$\begin{tikzcd} C(K) \arrow[r, "T"] & X \arrow[d, "U"] \\ c_0 \arrow[u, "V"] \arrow[r, "I_{c_0}"] & c_0 \end{tikzcd} \quad = \quad \begin{tikzcd} C(K) \arrow[r, "T"] & X \\ c_0 \arrow[u, "V"] \arrow[ur] & \end{tikzcd} \quad + \quad \begin{tikzcd} c_0 \arrow[r] & X \arrow[d, "U"] \\ & c_0 \end{tikzcd}$$

VT bounded below by (almost) $\|T\|_e/2$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$(\eta_n)_{n=1}^{\infty}$ disjoint $\qquad \|T\eta_n\| \gtrsim \frac{\|T\|_e}{2}$

An overview: quantitative factorization

$$\begin{tikzcd} C(K) \arrow[r, "T"] & X \\ c_0 \arrow[u, "V"] \arrow[ur] & \end{tikzcd}$$

$$\overbrace{(\eta_n)_{n=1}^{\infty} \text{ disjoint} \qquad \|T\eta_n\| \gtrsim \frac{\|T\|_e}{2}}^{VT \text{ bounded below by (almost) } \|T\|_e/2}$$

- Compact operators are nice to work with!
- $C(K)^* \cong \ell^1(K)$, nice space!

What is next?

Theorem (Pełczyński, 1962; factorization form by Arnott and Laustsen, 2023). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^*-sequentially compact, and let K be a compact Hausdorff space.

Then, for any non-weakly compact operator $T : C(K) \to X$, there exist $V \in \mathscr{B}(c_0, C(K))$ and $U \in \mathscr{B}(X, c_0)$ such that

$$UTV = I_{c_0}.$$

$$\begin{tikzcd} C(K) \arrow[r, "T"] & X \arrow[d, "U"] \\ c_0 \arrow[u, "V"] \arrow[r, "I_{c_0}"] & c_0 \end{tikzcd}$$

Definition. Given $T : C(K) \to X$ we define

$$\Delta(T) = \sup_{(f_n)} \limsup_{n\to\infty} \|Tf_n\|,$$

where the supremum is taken over all families of disjoint functions (f_n).

Remark: $\Delta(T) = 0$ if and only if T is weakly compact.

What is next?

Theorem ("Trivial" quantitative version). Let X be any Banach space not containing ℓ^1 or such that B_{X^*} is w^*-sequentially compact, and let K be a compact Hausdorff space.

Then, for any $\varepsilon > 0$ and any non-weakly compact operator $T : C(K) \to X$, there exist $V \in \mathscr{B}(c_0, C(K))$ and $U \in \mathscr{B}(X, c_0)$ such that

$$UTV = I_{c_0}, \quad \|U\| \, \|V\| < \frac{1+\varepsilon}{\Delta(T)}.$$

$$K = [0,1]$$

Question 1. Is $\Delta(T) \sim \|T\|_{we}$ for $K = [0,1]$?

Question 2. Is $\|T\|_{we}$ dual, i.e. does there exists c such that

$$\|T\|_{we} \geq \|T^*\|_{we} \geq c \, \|T\|_{we} \, .$$

Questions?

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