

LANCASTER UNIVERSITY



Banach and Metric Space Geometry Seminar

Complemented subspaces of L_1

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The complemented subspace problem in L_1

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Problem. Given a Banach space Z , describe its infinite-dimensional complemented subspaces up to isomorphism.

When Z is a “classical Banach space”: $c_0, \ell_p, C[0, 1], L_p, \dots$

Theorem (Pełczyński, 1960). *Every infinite-dimensional complemented subspace of c_0 is isomorphic to c_0 . For $1 \leq p < \infty$, every infinite-dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p .*

Theorem (Lindenstrauss, 1967). *Every infinite-dimensional complemented subspace of ℓ_∞ is isomorphic to ℓ_∞ .*

Question. Is every complemented subspace of $C[0, 1]$ isomorphic to $C(K)$ for some compact metrizable space K ?



For $1 < p < \infty$, $p \neq 2$, the familiar complemented subspaces of $L_p[0, 1]$ include

$$\ell_p, \ell_2, \ell_p \oplus \ell_2, \left(\bigoplus_{n=1}^{\infty} \ell_2 \right)_{\ell_p}, L_p[0, 1].$$

For $p > 2$, Rosenthal (1970) and Schechtman (1975) constructed further examples.

Theorem (Bourgain–Rosenthal–Schechtman, 1981). *The space $L_p[0, 1]$ has uncountably many pairwise nonisomorphic complemented subspaces.*

They have unconditional bases, hence are isomorphic to Banach lattices.

Question (Lindenstrauss–Rosenthal, 1969). *Is every infinite-dimensional complemented subspace of $L_1[0, 1]$ isomorphic to ℓ_1 or $L_1[0, 1]$?*

For such a subspace X :

Theorem (Lindenstrauss–Pełczyński, 1968). *If X has an unconditional basis, then $X \simeq \ell_1$.*

Theorem (Lewis–Stegall, 1973). *If X has the Radon–Nikodým property, then $X \simeq \ell_1$.*

Theorem (Enflo–Starbird, 1979). *If X contains a copy of $L_1[0, 1]$, then $X \simeq L_1[0, 1]$.*

Let X be an infinite-dimensional complemented subspace of $L_1[0, 1]$.

Questions (Popov, 2020).

(a) If X has the Schur property, must $X \simeq \ell_1$?

Theorem A. *There exists an infinite-dimensional complemented subspace of L_1 which has the Schur property and is not isomorphic to ℓ_1 .*

(b) If X does not have the Schur property, must $X \simeq L_1[0, 1]$?

Theorem B. *There exists an infinite-dimensional complemented subspace of L_1 which does not have the Schur property and is not isomorphic to L_1 .*

Theorem C. *There is a family $(X_A)_{A \in \mathcal{A}}$, $|\mathcal{A}| = \mathfrak{c}$, of complemented subspaces of $L_1[0, 1]$. Each X_A has the Schur property but fails the Radon–Nikodým property. If $A \neq B$, then X_A is not isomorphic to a subspace of X_B .*

Corollary. *The algebra $\mathcal{L}(L_1)$ contains $2^{\mathfrak{c}}$ distinct proper closed two-sided ideals.*



Contractive C_0 -semigroups
on X

→

Complemented subspaces of
 $\ell_p(\mathbb{N}_0; X)$, $1 \leq p < \infty$

Definition. On a Banach space X , a *contractive C_0 -semigroup* is a family $u \mapsto C_u$ ($u \geq 0$) of bounded linear maps $C_u : X \rightarrow X$ such that

- i) $C_0 = I_X$ and $C_{u+v} = C_u C_v$ for all $u, v \geq 0$,
- ii) $\|C_u\| \leq 1$ for all $u \geq 0$.
- iii) For every $f \in X$, $\lim_{u \downarrow 0} \|C_u f - f\|_X = 0$.

Examples.

$$C_u = I_X \quad \text{for all } u \geq 0, \quad C_u = e^{-u} I_X \quad \text{for all } u \geq 0.$$

For $t > 0$, average the semigroup against an exponential density:

$$R_t f = \frac{1}{t} \int_0^\infty e^{-u/t} C_u f \, du.$$

Each R_t is a contraction, and $R_t f \rightarrow f$ in X as $t \downarrow 0$.

Resolvent identity.

$$R_s R_t = \frac{tR_t - sR_s}{t - s} \quad (s, t > 0, s \neq t)$$

For the infinitesimal generator A , $R_t = (I - tA)^{-1}$, so

$$tR_t - sR_s = R_s [t(I - sA) - s(I - tA)] R_t = (t - s)R_s R_t. \quad \square$$



Fix $1 \leq p < \infty$. Then $R_{2^{-n}} \xrightarrow[n \rightarrow \infty]{\text{strong operator}} I$. sol $I \rightarrow 0$

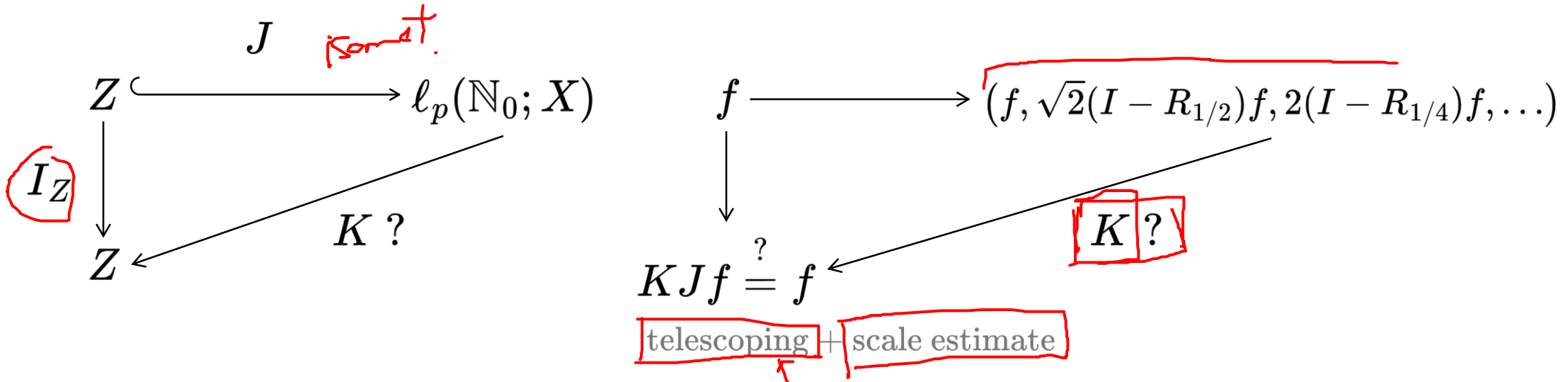
For $f \in X$, define

$$\|f\|_Z^p = \|f\|_X^p + \sum_{n \geq 1} 2^{np/2} \|(I - R_{2^{-n}})f\|_X^p.$$

$$Z = Z_{C,p}(X) = \{f \in X : \|f\|_Z < \infty\}.$$

$$R_s R_t = \frac{tR_t - sR_s}{t - s}$$

$(s, t > 0, s \neq t)$





Lemma. For $s, t > 0$,

- i) $\|(I - R_s)R_t\| \leq 2 \min\{1, s/t\}$
- ii) $R_{2t}(I - R_t) = R_t - R_{2t}$

scale estimate

telescoping

$$R_s R_t = \frac{tR_t - sR_s}{t - s}$$

$(s, t > 0, s \neq t)$

Proof. i) By the resolvent identity,

$$(I - R_s)R_t = \frac{s}{t}R_s(I - R_t) \xrightarrow{\text{contractivity}} \|(I - R_s)R_t\| \leq \min\{2, 2s/t\}.$$

ii) Taking $s = 2t$,

$$R_{2t}R_t = 2R_{2t} - R_t. \quad \square$$

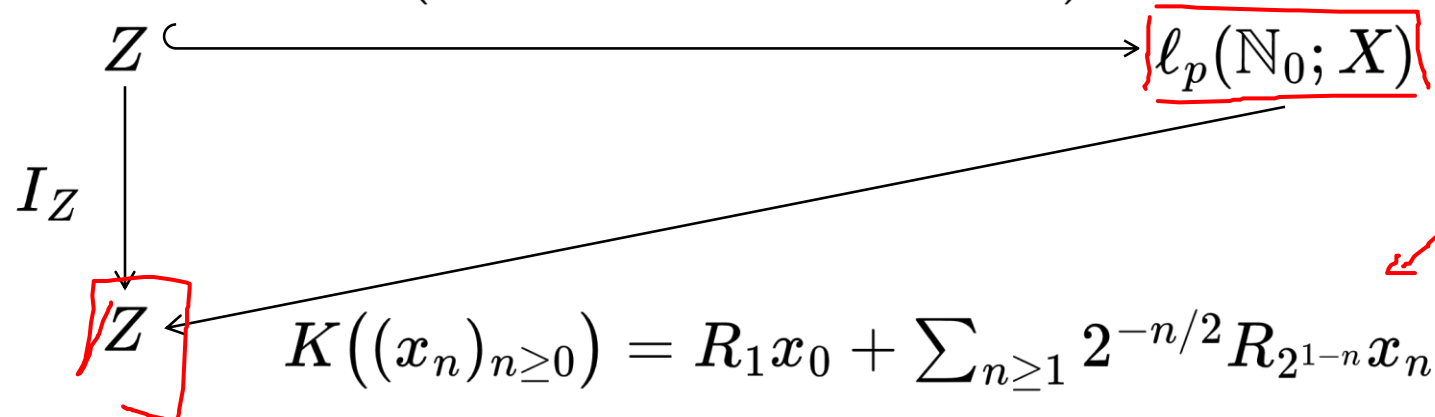


Theorem 1. For $1 \leq p < \infty$, $Z = Z_{C,p}(X)$ embeds isometrically as a complemented subspace of $\ell_p(\mathbb{N}_0; X)$.

Proof sketch. Recall that

$$Z = \left\{ f \in X : \sum_{n \geq 1} 2^{np/2} \|(I - R_{2^{-n}})f\|_X^p < \infty \right\}.$$

$$Jf = (f, (2^{n/2}(I - R_{2^{-n}})f)_{n \geq 1})$$



$$\begin{aligned} \text{i) } & \|(I - R_s)R_t\| \\ & \leq 2 \min\{1, s/t\} \end{aligned}$$

$$\begin{aligned} \text{ii) } & R_{2t}(I - R_t) \\ & = R_t - R_{2t} \end{aligned}$$



$$J = \begin{pmatrix} I \\ \sqrt{2}(I - R_{1/2}) \\ 2(I - R_{1/4}) \\ \vdots \end{pmatrix}, \quad K = (R_1 \quad 2^{-1/2}R_1 \quad \dots).$$

$$J_N f = ((Jf)_0, \dots, (Jf)_N, 0, \dots).$$

$$K J_N f = R_1 f + \sum_{n=1}^N R_{2^{1-n}} (I - R_{2^{-n}}) f$$

$$\text{ii) } = R_1 f + \sum_{n=1}^N (R_{2^{-n}} - R_{2^{1-n}}) f$$

$$= R_{2^{-N}} f \xrightarrow[N \rightarrow \infty]{\text{strong continuity}} f.$$

$$\begin{aligned} \text{i) } & \| (I - R_s) R_t \| \\ & \leq 2 \min\{1, s/t\} \end{aligned}$$

$$\begin{aligned} \text{ii) } & R_{2t}(I - R_t) \\ & = R_t - R_{2t} \end{aligned}$$

$$K J_N \xrightarrow[N \rightarrow \infty]{} I$$



K is bounded. For finitely supported x ,

$$\boxed{K}x = \begin{pmatrix} R_1 & 2^{-1/2}R_1 & \dots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \end{pmatrix}.$$

$$(\boxed{JK})_{ij} = 2^{(i-j)/2} \overbrace{(I - R_{2^{-i}})R_{2^{1-j}}} \quad (i, j \geq 1).$$

$$\xrightarrow{\text{i)}} \quad \|(JK)_{ij}\| \leq 2 \cdot 2^{-|i-j|/2}.$$

$$\|(JKx)_i\|_X \leq 2 \sum_{j \geq 0} 2^{-|i-j|/2} \|x_j\|_X.$$

$$\|\boxed{K}x\|_Z = \|JKx\|_{\ell_p(\mathbb{N}_0; X)} \stackrel{\text{Young}}{\lesssim} \|x\|_{\ell_p(\mathbb{N}_0; X)}.$$

$$\begin{aligned} \text{i)} \quad & \|(I - R_s)R_t\| \\ & \leq 2 \min\{1, s/t\} \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad & R_{2t}(I - R_t) \\ & = R_t - R_{2t} \end{aligned}$$



$P = JK$ is a bounded projection.

$\leftarrow I = I$

$$J_N f \longrightarrow Jf \quad \text{in } \ell_p(\mathbb{N}_0; X).$$

$$KJ_N f = R_{2^{-N}} f \longrightarrow \begin{cases} KJf & \text{in } Z \quad (K \text{ bounded}), \\ f & \text{in } X \quad (\text{strong continuity}). \end{cases}$$

$$\Downarrow \\ \boxed{KJ = I_Z.}$$

$$P^2 = J(KJ)K = P, \quad \text{ran } P = JZ. \quad \square$$

C_0 -semigroup
on X

complemented subspace of
 $\ell_p(\mathbb{N}_0; X)$

$$\begin{aligned} \text{i) } & \| (I - R_s)R_t \| \\ & \leq 2 \min\{1, s/t\} \end{aligned}$$

$$\begin{aligned} \text{ii) } & R_{2t}(I - R_t) \\ & = R_t - R_{2t} \end{aligned}$$



Theorem 1. For a contractive C_0 -semigroup on X and $1 \leq p < \infty$, $Z_{C,p}(X)$ embeds isometrically as a complemented subspace of $\ell_p(\mathbb{N}_0; X)$.

Take $p = 1$ and $X = L_1(G, m)$, where

$$G = \{-1, 1\}^{\mathbb{N}}, \quad m = \bigotimes_{j \geq 1} \frac{1}{2}(\delta_{-1} + \delta_1).$$

The multiplication is coordinatewise, and m is Haar probability.

$$Z_C := Z_{C,1}(L_1(G, m)) \xhookrightarrow{J} \ell_1(\mathbb{N}_0; L_1(G, m)) \cong L_1.$$

Which semigroup gives an interesting complemented subspace?

Theorem B. There exists a complemented subspace of L_1 which does not have the Schur property and is not isomorphic to L_1 .

We seek a bounded projection $P : L_1 \rightarrow L_1$ with:

1. A non-Schur range:

$$\underbrace{P \text{ is not Dunford-Pettis}} \quad \overset{\text{Rosenthal}}{\iff} \quad \boxed{\ell_2 \hookrightarrow \text{ran } P.}$$

2. P is non-Enflo: its range contains no copy of L_1 .

Rosenthal's biased-coin operators fix a copy of ℓ_2 but no copy of L_1 . They are not projections.



Rosenthal's biased coins. For $0 < \varepsilon < 1$, set

$$\eta_\varepsilon = \bigotimes_{j \geq 1} \left(\frac{1 + \varepsilon}{2} \delta_1 + \frac{1 - \varepsilon}{2} \delta_{-1} \right).$$

$$T_\varepsilon f(x) := \int_G f(xy^{-1}) d\eta_\varepsilon(y) = (f * \eta_\varepsilon)(x).$$

$$\int_G y_j d\eta_\varepsilon(y) = \varepsilon \quad (j \geq 1).$$

Thus ε is the mean of each sign.



Two properties (Rosenthal).

i) For $\underbrace{r_j(x)} = x_j$,

$$T_\varepsilon r_j = \underbrace{\varepsilon}_{\swarrow} r_j, \quad r_j \xrightarrow{w} 0 \text{ in } L_1, \quad \|T_\varepsilon r_j\|_1 = \varepsilon > 0.$$

In fact, T_ε fixes a copy of ℓ_2 , so it is not Dunford–Pettis.

$$T \Big|_{\text{span}\{r_j\}} = \sum I$$

ii) T_ε is *non-Enflo*: it fixes no copy of L_1 , meaning it is bounded below on no subspace isomorphic to L_1 .

These are the properties we would like a projection to have.

But T_ε is not a projection:

$$\overbrace{T_\varepsilon^2 r_j} = \varepsilon^2 r_j \neq \varepsilon r_j = T_\varepsilon r_j.$$



We have the following composition law:

$$\boxed{T_\varepsilon T_\delta = T_{\varepsilon\delta}} \quad (0 < \varepsilon, \delta < 1).$$

For additive time, put

$$C_u := \boxed{T_{e^{-u}}} \quad (u \geq 0), \quad C_0 = I.$$

$$\boxed{C_u C_v = T_{e^{-(u+v)}} = C_{u+v}.$$

$$\mu_{\underline{u}} := \eta_{e^{-u}} = \bigotimes_{j \geq 1} \left(\frac{1 + \boxed{e^{-u}}}{2} \delta_1 + \frac{1 - \boxed{e^{-u}}}{2} \delta_{-1} \right), \quad C_u f = f * \mu_u.$$

Lemma. $(C_u)_{u \geq 0}$ is a positive contractive C_0 -semigroup on $L_1(G, m)$.



By Theorem 1, Z_C is a complemented subspace of L_1 .

$$\rightarrow R_t f = \frac{1}{t} \int_0^\infty e^{-u/t} C_u f du.$$

$$C_u r_j = e^{-u} r_j \implies R_t r_j = \frac{1}{1+t} r_j. \implies j \in \mathbb{Z}_t$$

For finite Rademacher sums,

$$\left\| \sum_j a_j r_j \right\|_{Z_C} = \gamma \left\| \sum_j a_j r_j \right\|_1 \underset{\text{Khintchine}}{\asymp} \left(\sum_j |a_j|^2 \right)^{1/2}$$

$$\|f\|_{Z_C} = \|f\|_1 + \sum_{n \geq 1} 2^{n/2} \|(I - R_{2^{-n}})f\|_1.$$

where $\gamma = 1 + \sum_{n \geq 1} \frac{2^{-n/2}}{1+2^{-n}} < \infty$.

Thus Z_C contains a copy of ℓ_2 and therefore, in particular, is non-Schur.



Handwritten notes and diagram:

- $P: \ell_1(L) \rightarrow \ell_1(L)$
- $P = JK =$ (where P is circled and J, K are underlined)
- Matrix representation of P (with P_{11} boxed):

$$\begin{pmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & \boxed{P_{11}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$
- Formula for P_{ij} (with P_{ij} boxed):

$$\boxed{P_{ij}} = 2^{(i-j)/2} (I - R_{2^{-i}}) R_{2^{1-j}} \quad (i, j \geq 1).$$

Theorem (Liu, 1998). In $\mathcal{L}(L_1)$, the following are bands:

- i) non-Enflo operators;
- ii) Dunford–Pettis operators.

Put $H_N(x_n) = (x_0, \dots, x_N, 0, \dots)$.

Lemma. Let $\mathcal{B}$ be a band in $\mathcal{L}(\ell_1(\mathbb{N}_0; L_1))$ and T bounded. Then

$$[H_N T H_N \in \mathcal{B} \text{ for every } N] \implies T \in \mathcal{B}.$$

Proof. $|H_N T H_N| = H_N |T| H_N \uparrow |T|.$ $\square$

The resolvents are non-Enflo. For $a > 0$, put

$$R_{t,a} := \frac{1}{t} \int_a^\infty e^{-u/t} C_u du = e^{-a/t} C_a R_t.$$

Rosenthal's theorem makes $C_a = T_{e^{-a}}$ non-Enflo. Bounded composition gives the same property for $R_{t,a}$.

$$\|R_t - R_{t,a}\|_{1 \rightarrow 1} \leq 1 - e^{-a/t} \rightarrow 0 \quad (a \downarrow 0).$$

By norm closure, R_t is non-Enflo.

$$R_t = \frac{1}{t} \int_0^\infty e^{-u/t} C_u du$$

$$\|R_t - R_{t,a}\| =$$

The projection is non-Enflo.

$$P = JK = \begin{pmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad P_{ij} = 2^{(i-j)/2} (I - R_{2^{-i}}) R_{2^{1-j}} \quad (i, j \geq 1).$$

Each block is a bounded composition with a resolvent, hence non-Enflo.

Every $H_N P H_N$ is non-Enflo. By the band lemma, so is P .

$$P|_{JZ_C} = I \implies Z_C \text{ contains no copy of } L_1. \quad \square$$

Theorem B. *There is a complemented $X \subseteq L_1$ containing ℓ_2 but no copy of L_1 .*



Measures



Semigroup



Theorem 1

**Complemented
subspace**

$$\mu_u = \eta_{e^{-u}} = \bigotimes_{j \geq 1} \left(\frac{1 + e^{-u}}{2} \delta_1 + \frac{1 - e^{-u}}{2} \delta_{-1} \right)$$

$$\mathbb{E}_{\mu_u}[y_j] = e^{-u}$$

$$C_u f(x) = \int_G f(xy^{-1}) d\mu_u(y) = (f * \mu_u)(x) \leftarrow$$

$$R_t = \int_0^\infty t^{-1} e^{-u/t} C_u du$$

$$\|f\|_{Z_C} = \|f\|_1 + \sum_{n \geq 1} 2^{n/2} \|(I - R_{2^{-n}})f\|_1$$

$$Z_C \xrightarrow{J} \ell_1(\mathbb{N}_0; L_1) \cong L_1, \quad KJ = I$$

$$\ell_2 \hookrightarrow Z_C$$

$$L_1 \not\hookrightarrow Z_C$$

$$C_u|_{\text{span}\{r_j\}} = e^{-u} I$$

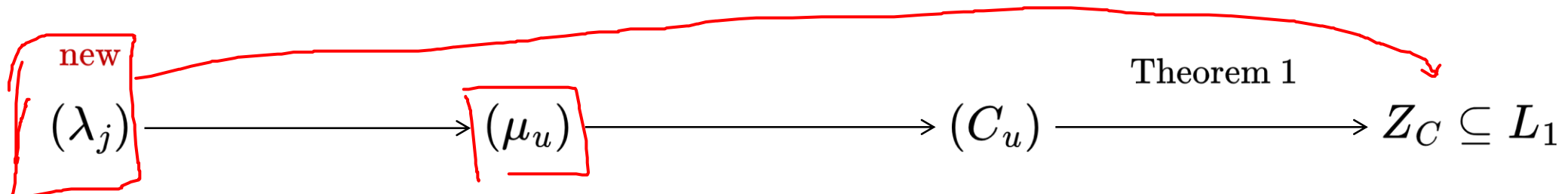
Rosenthal + cutoff + Liu

Change the rates: e^{-u} $\longrightarrow$ $e^{-u\lambda_j}$, with $\lambda_j > 0$. $\lambda_j > 0$

$$\mu_u = \bigotimes_{j \geq 1} \left(\frac{1 + e^{-u\lambda_j}}{2} \delta_1 + \frac{1 - e^{-u\lambda_j}}{2} \delta_{-1} \right).$$

$$C_u f(x) := \int_G f(xy^{-1}) d\mu_u(y) = (f * \mu_u)(x).$$

Lemma. For every sequence $(\lambda_j)_{j \geq 1}$ with $\lambda_j > 0$, $(C_u)_{u \geq 0}$ is a positive contractive C_0 -semigroup on $L_1(G, m)$.



Theorem A. *There exists an infinite-dimensional complemented subspace of L_1 which has the Schur property and is not isomorphic to ℓ_1 .*

1. P is Dunford–Pettis, so Z_C is Schur.

$$\lambda_j \rightarrow \infty \xRightarrow{\text{Walsh eigenvalues}} \frac{1}{1 + t \sum_{j \in F} \lambda_j} \rightarrow 0 \implies R_t \text{ compact on } L_2.$$

$$R_t \text{ compact on } L_2 \implies \boxed{R_t|_{L_\infty}} \text{ compact} \implies P_{ij}|_{L_\infty} \text{ compact.}$$

$$P_{ij}|_{L_\infty} \text{ compact} \xRightarrow{\text{Girardi / Dunford–Pettis equi-integrability criterion}} \boxed{P_{ij} \text{ Dunford–Pettis.}}$$

$$P_{ij} \text{ Dunford–Pettis} \xRightarrow{\text{Liu + band lemma}} P \text{ Dunford–Pettis} \implies Z_C \text{ Schur.}$$



2. Z_C fails RNP, so $Z_C \not\cong \ell_1$.

$$R_1 f = f * \nu_1 = K(f, 0, \dots), \quad \nu_1 = \int_0^\infty e^{-u} \mu_u du.$$

$$\text{slow growth} \xrightarrow{\text{Chebyshev} + \text{Borel-Cantelli}} \begin{matrix} m(N)=0 \\ \mu_u(N)=1 \ (u>0) \end{matrix} \implies \nu_1 \perp m.$$

$$\nu_1 \perp m \xrightarrow{\text{Costé}} R_1 : L_1 \rightarrow L_1 \text{ not representable.}$$

$$R_1 : L_1 \rightarrow L_1 \text{ not representable} \xrightarrow[\text{composition with } Z_C \hookrightarrow L_1]{} R_1 : L_1 \rightarrow Z_C \text{ not representable.}$$

RNP would make every bounded operator $L_1 \rightarrow Z_C$ representable.

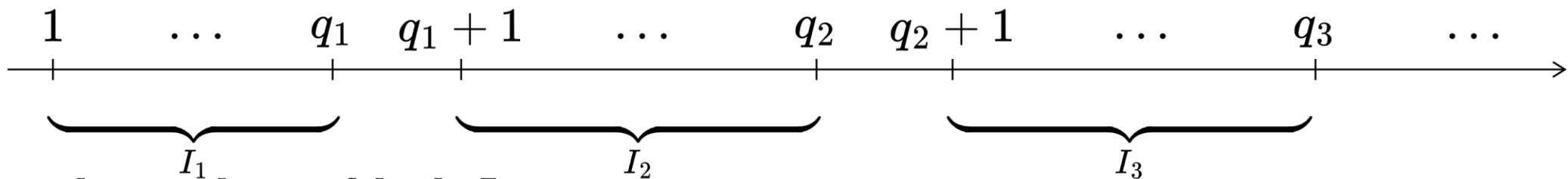
$$\implies Z_C \text{ fails RNP} \implies Z_C \not\cong \ell_1.$$

We balance $\lambda_j \rightarrow \infty$ with slow growth: $\lambda_j = \sqrt{\log(j+1)}$.



Theorem C. *There are $\mathfrak{c}$ complemented subspaces $(X_A)_{A \in \mathcal{A}}$ of L_1 , each Schur and without RNP, with no embeddings between distinct members.*

Choose an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$.



On each coordinate block I_s , put

$$\lambda_j^A = \begin{cases} \ell_s, & \underline{s \in A}, \\ h_s, & s \notin A, \end{cases} \quad j \in I_s, \quad 0 < \ell_s < h_s < \ell_{s+1}, \quad \ell_s \rightarrow \infty.$$

For $A \neq B$, there are infinitely many blocks with

$$s \in A \setminus B : \quad \underbrace{\lambda_j^A = \ell_s}_{\text{slow}} \quad \text{and} \quad \underbrace{\lambda_j^B = h_s}_{\text{fast}} \quad (j \in I_s).$$

Theorem 1. *Let (C_u) be a contractive C_0 -semigroup on a Banach space X , and let $1 \leq p < \infty$. Then $Z_{C,p}(X)$ is isometric to a complemented subspace of $\ell_p(\mathbb{N}_0; X)$.*

For $X = L_p[0, 1]$, put

$$Z_{C,p} := Z_{C,p}(L_p[0, 1]).$$

$$Z_{C,p} \xhookrightarrow{J} \ell_p(\mathbb{N}_0; L_p[0, 1]) \cong L_p[0, 1], \quad KJ = I.$$

Question. Which complemented subspaces of L_p are isomorphic to $Z_{C,p}$?

Current work. *Let $1 \leq p < \infty$. The spaces $Z_{C,p}$, for uniformly bounded C_0 -semigroups on $L_p[0, 1]$, are, up to isomorphism, precisely*

$$\left\{ \begin{array}{l} \text{complemented subspaces of } L_p[0, 1] \\ \text{containing a complemented copy of } \ell_p \end{array} \right\}.$$

For $p = 1$, this includes every infinite-dimensional complemented subspace of L_1 .

Proof of the converse. On $Z \oplus_p L_p \simeq L_p$, take $C_u = I_Z \oplus T_u$, where the fast dyadic semigroup satisfies $Z_{T,p} \simeq \ell_p$. Then

$$Z_{C,p} \simeq Z \oplus_p Z_{T,p} \simeq Z \oplus_p \ell_p \simeq Z. \quad \square$$



Question. Which rates give isomorphic spaces?

$$Z_{C,p} \simeq Z_{D,p}$$

Let K be compact metrizable.

Question. *Is every complemented subspace of $C(K)$ isomorphic to $C(L)$ for some compact metrizable space L ?*



Thanks!

Questions?